Lie groups as four-dimensional special complex manifolds with Norden metric

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Abstract

An example of a four-dimensional special complex manifold with Norden metric of constant holomorphic sectional curvature is constructed via a two-parametric family of solvable Lie algebras. The curvature properties of the obtained manifold are studied. Necessary and sufficient conditions for the manifold to be isotropic Kählerian are given.
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1 Preliminaries

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that:

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y), \qquad x, y \in \mathfrak{X}(M). \tag{1.1}$$

The associated metric $\widetilde{g}$ of g on M, given by $\widetilde{g}(x, y) = g(x, Jy)$, is a Norden metric, too. Both metrics are necessarily neutral, i.e. of signature (n, n).

If ∇ is the Levi-Civita connection of g, the tensor field F of type $(0, 3)$ is defined by

$$F(x, y, z) = g\left((\nabla_x J)y, z\right) \tag{1.2}$$

and has the following symmetries

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz). \tag{1.3}$$

Let $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$. The Lie 1-forms θ and θ^* associated with F are defined by, respectively

$$\theta(x) = g^{ij}F(e_i, e_j, x), \qquad \theta^* = \theta \circ J. \tag{1.4}$$

The Nijenhuis tensor field N for J is given by

$$N(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, Jy]. \tag{1.5}$$

It is known [4] that the almost complex structure is complex iff it is integrable, i.e. $N = 0$.

A classification of the almost complex manifolds with Norden metric is introduced in [2], where eight classes of these manifolds are characterized according to the properties of F. The three basic classes: $\mathcal{W}_1$, $\mathcal{W}_2$ of *the special complex manifolds with Norden metric* and $\mathcal{W}_3$ of *the quasi-Kähler manifolds with Norden metric* are given as follows:

$$
\begin{aligned}
&\mathcal{W}_1 : F(x,y,z) = \tfrac{1}{2n}\left[g(x,y)\theta(z) + g(x,z)\theta(y)\right.\\
&\qquad\qquad\left.+g(x,Jy)\theta(Jz) + g(x,Jz)\theta(Jy)\right];\\
&\mathcal{W}_2 : F(x,y,Jz) + F(y,z,Jx) + F(z,x,Jy) = 0, \quad \theta = 0 \;\Leftrightarrow\; N = 0, \quad \theta = 0;\\
&\mathcal{W}_3 : F(x,y,z) + F(y,z,x) + F(z,x,y) = 0.
\end{aligned}
\tag{1.6}
$$

The class $\mathcal{W}_0$ of *the Kähler manifolds with Norden metric* is defined by $F = 0$ and is contained in each of the other classes.

Let R be the curvature tensor of ∇, i.e. $R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The corresponding (0,4)-type tensor is defined by $R(x,y,z,u) = g\left(R(x,y)z,u\right)$. The Ricci tensor ρ and the scalar curvatures τ and τ^* are given by:

$$
\rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad \tau = g^{ij}\rho(e_i,e_j), \qquad \tau^* = g^{ij}\rho(e_i,Je_j).
\tag{1.7}
$$

A tensor of type (0,4) is said to be *curvature-like* if it has the properties of R. Let S be a symmetric (0,2)-tensor. We consider the following curvature-like tensors:

$$
\begin{aligned}
\psi_1(S)(x,y,z,u) &= g(y,z)S(x,u) - g(x,z)S(y,u)\\
&\quad + g(x,u)S(y,z) - g(y,u)S(x,z),\\
\pi_1 = \tfrac{1}{2}\psi_1(g), &\quad \pi_2(x,y,z,u) = g(y,Jz)g(x,Ju) - g(x,Jz)g(y,Ju).
\end{aligned}
\tag{1.8}
$$

It is known that on a pseudo-Riemannian manifold M ($\dim M = 2n \geq 4$) the conformal invariant Weyl tensor has the form

$$
W(R) = R - \frac{1}{2(n-1)}\big\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\big\}.
\tag{1.9}
$$

Let $\alpha = \{x,y\}$ be a non-degenerate 2-plane spanned by the vectors $x,y \in T_pM$, $p \in M$. The sectional curvature of α is given by

$$
k(\alpha;p) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}.
\tag{1.10}
$$

We consider the following basic sectional curvatures in T_pM with respect to the structures J and g: *holomorphic sectional curvatures* if $J\alpha = \alpha$ and *totally real sectional curvatures* if $J\alpha \perp \alpha$ with respect to g.

The square norm of ∇J is defined by $\|\nabla J\|^2 = g^{ij}g^{kl}g\big((\nabla_{e_i}J)e_k, (\nabla_{e_j}J)e_l\big)$. Then, by (1.2) we get

$$
\|\nabla J\|^2 = g^{ij}g^{kl}g^{pq}F_{ikp}F_{jlq},
\tag{1.11}
$$

where $F_{ikp} = F(e_i,e_k,e_p)$.

An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is called an *isotropic Kähler manifold with Norden metric* [3].

2 Almost complex manifolds with Norden metric of constant holomorphic sectional curvature

In this section we obtain a relation between the vanishing of the holomorphic sectional curvature and the vanishing of $\|\nabla J\|^2$ on $\mathcal{W}_2$-manifolds and $\mathcal{W}_3$-manifolds with Norden metric.

In [1] it is proved the following

Theorem A. ([1]) *An almost complex manifold with Norden metric is of pointwise constant holomorphic sectional curvature if and only if*

$$\begin{aligned}
&3\{R(x,y,z,u)+R(x,y,Jz,Ju)+R(Jx,Jy,z,u)+R(Jx,Jy,Jz,Ju)\}\\
&-R(Jy,Jz,x,u)+R(Jx,Jz,y,u)-R(y,z,Jx,Ju)+R(x,z,Jy,Ju)\\
&-R(Jx,z,y,Ju)+R(Jy,z,x,Ju)-R(x,Jz,Jy,u)+R(y,Jz,Jx,u)\\
&=8H\{\pi_1+\pi_2\}
\end{aligned} \tag{2.1}$$

for some $H \in FM$ and all $x,y,z,u \in \mathfrak{X}(M)$. In this case $H(p)$ is the holomorphic sectional curvature of all holomorphic non-degenerate 2-planes in T_pM, $p \in M$. ■

Taking into account (1.7) and (1.8), the total trace of (2.1) implies

$$H(p)=\frac{1}{4n^2}(\tau+\tau^{**}), \tag{2.2}$$

where $\tau^{**}=g^{il}g^{jk}R(e_i,e_j,Je_k,Je_l)$.

In [5] we have proved that on a $\mathcal{W}_2$-manifold it is valid

$$\|\nabla J\|^2=2(\tau+\tau^{**}), \tag{2.3}$$

and in [3] it is proved that on a $\mathcal{W}_3$-manifold

$$\|\nabla J\|^2=-2(\tau+\tau^{**}). \tag{2.4}$$

Then, by Theorem A, (2.2), (2.3) and (2.4) we obtain

Theorem 2.1. *Let (M,J,g) be an almost complex manifold with Norden metric of pointwise constant holomorphic sectional curvature $H(p)$, $p \in M$. Then*

(i) $\|\nabla J\|^2=8n^2H(p)$ *if* $(M,J,g)\in\mathcal{W}_2$;

(ii) $\|\nabla J\|^2=-8n^2H(p)$ *if* $(M,J,g)\in\mathcal{W}_3$. ■

Theorem 2.1 implies

Corollary 2.2. *Let (M,J,g) be a $\mathcal{W}_2$-manifold or $\mathcal{W}_3$-manifold of pointwise constant holomorphic sectional curvature $H(p)$, $p\in M$. Then, (M,J,g) is isotropic Kählerian iff $H(p)=0$.*

In the next section we construct an example of a $\mathcal{W}_2$-manifold of constant holomorphic sectional curvature.

3 Lie groups as four-dimensional $\mathcal{W}_2$-manifolds

Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra corresponding to a real connected Lie group G. If $\{X_1, X_2, X_3, X_4\}$ is a basis of left invariant vector fields on G and $[X_i, X_j] = C_{ij}^k X_k$ $(i, j, k = 1, 2, 3, 4)$ then the structural constants C_{ij}^k satisfy the anti-commutativity condition $C_{ij}^k = -C_{ji}^k$ and the Jacobi identity $C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^l + C_{si}^k C_{kj}^l = 0$.

We define an almost complex structure J and a compatible metric g on G by the conditions, respectively:

$$JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2, \tag{3.1}$$

$$\begin{aligned} &g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1, \\ &g(X_i, X_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4. \end{aligned} \tag{3.2}$$

Because of (1.1), (3.1) and (3.2) g is a Norden metric. Thus, (G, J, g) is a 4-dimensional almost complex manifold with Norden metric.

From (3.2) it follows that the well-known Levi-Civita identity for g takes the form

$$2g(\nabla_{X_i} X_j, X_k) = g([X_i, X_j], X_k) + g([X_k, X_i], X_j) + g([X_k, X_j], X_i). \tag{3.3}$$

Let us denote $F_{ijk} = F(X_i, X_j, X_k)$. Then, by (1.2) and (3.3) we have

$$\begin{aligned} 2F_{ijk} = g\big([X_i, JX_j] - J[X_i, X_j], X_k\big) + g\big(J[X_k, X_i] - [JX_k, X_i], X_j\big) \\ +g\big([X_k, JX_j] - [JX_k, X_j], X_i\big). \end{aligned} \tag{3.4}$$

According to (1.6) to construct an example of a $\mathcal{W}_2$-manifold we need to find sufficient conditions for the Nijenhuis tensor N and the Lie 1-form θ to vanish on $\mathfrak{g}$.

By (1.2), (1.5), (3.2) and (3.4) we compute the essential components N_{ij}^k $(N(X_i, X_j) = N_{ij}^k X_k)$ of N and $\theta_i = \theta(X_i)$ of θ, respectively, as follows:

$$\begin{aligned} &N_{12}^1 = C_{34}^1 - C_{12}^1 - C_{23}^3 + C_{14}^3, \qquad &&\theta_1 = 2C_{13}^1 - C_{12}^4 + C_{14}^2 + C_{23}^2 - C_{34}^4, \\ &N_{12}^2 = C_{34}^2 - C_{12}^2 - C_{23}^4 + C_{14}^4, &&\theta_2 = 2C_{24}^2 + C_{12}^3 + C_{14}^1 + C_{23}^1 + C_{34}^3, \\ &N_{12}^3 = C_{34}^3 - C_{12}^3 + C_{23}^1 - C_{14}^1, &&\theta_3 = 2C_{13}^3 + C_{12}^2 + C_{14}^4 + C_{23}^4 + C_{34}^2, \\ &N_{12}^4 = C_{34}^4 - C_{12}^4 + C_{23}^2 - C_{14}^2, &&\theta_4 = 2C_{24}^4 - C_{12}^1 + C_{14}^3 + C_{23}^3 - C_{34}^1. \end{aligned} \tag{3.5}$$

Then, (1.6) and (3.5) imply

Theorem 3.1. *Let (G, J, g) be a 4-dimensional almost complex manifold with Norden metric defined by (3.1) and (3.2). Then, (G, J, g) is a $\mathcal{W}_2$-manifold iff for the Lie algebra $\mathfrak{g}$ of G are valid the conditions*:

$$\begin{aligned} &C_{13}^1 = C_{12}^4 - C_{23}^2 = C_{34}^4 - C_{14}^2, \qquad &&C_{13}^3 = -\left(C_{12}^2 + C_{23}^4\right) = -\left(C_{14}^4 + C_{34}^2\right), \\ &C_{24}^4 = C_{12}^1 - C_{14}^3 = C_{34}^1 - C_{23}^3, &&C_{24}^2 = -\left(C_{12}^3 + C_{14}^1\right) = -\left(C_{23}^1 + C_{34}^3\right), \end{aligned} \tag{3.6}$$

where C_{ij}^k $(i, j, k = 1, 2, 3, 4)$ satisfy the Jacodi identity. ∎

One solution to (3.6) and the Jacobi identity is the 2-parametric family of solvable Lie algebras $\mathfrak{g}$ given by

$$
(3.7)\qquad \mathfrak{g}:\ \begin{array}{ll}
[X_1,X_2]=\lambda X_1-\lambda X_2, & [X_2,X_3]=\mu X_1+\lambda X_4,\\
[X_1,X_3]=\mu X_2+\lambda X_4, & [X_2,X_4]=\mu X_1+\lambda X_3,\\
[X_1,X_4]=\mu X_2+\lambda X_3, & [X_3,X_4]=-\mu X_3+\mu X_4, \qquad \lambda,\mu\in\mathbb{R}.
\end{array}
$$

Let us study the curvature properties of the $\mathcal{W}_2$-manifold (G,J,g), where the Lie algebra $\mathfrak{g}$ of G is defined by (3.7).

By (3.2), (3.3) and (3.7) we obtain the components of the Levi-Civita connection:

$$
(3.8)\qquad \begin{array}{ll}
\nabla_{X_1}X_2=\lambda X_1+\mu(X_3+X_4), & \nabla_{X_2}X_1=\lambda X_2+\mu(X_3+X_4),\\
\nabla_{X_3}X_4=-\lambda(X_1+X_2)-\mu X_3, & \nabla_{X_4}X_3=-\lambda(X_1+X_2)-\mu X_4,\\
\nabla_{X_1}X_1=-\lambda X_2,\quad \nabla_{X_2}X_2=-\lambda X_1, & \nabla_{X_3}X_3=\mu X_4,\quad \nabla_{X_4}X_4=\mu X_3,\\
\nabla_{X_1}X_3=\nabla_{X_1}X_4=\mu X_2, & \nabla_{X_2}X_3=\nabla_{X_2}X_4=\mu X_1,\\
\nabla_{X_3}X_1=\nabla_{X_3}X_2=-\lambda X_4, & \nabla_{X_4}X_1=\nabla_{X_4}X_2=-\lambda X_3.
\end{array}
$$

Taking into account (3.4) and (3.7) we compute the essential non-zero components of F:

$$
(3.9)\qquad \begin{array}{l}
F_{114}=-F_{214}=F_{312}=\frac{1}{2}F_{322}=\frac{1}{2}F_{411}=F_{412}=-\lambda,\\
F_{112}=\frac{1}{2}F_{122}=\frac{1}{2}F_{211}=F_{212}=-F_{314}=F_{414}=\mu.
\end{array}
$$

The other non-zero components of F are obtained from (1.3).

By (1.11) and (3.9) for the square norm of ∇J we get

$$
(3.10)\qquad \|\nabla J\|^2=-32(\lambda^2-\mu^2).
$$

Further, we obtain the essential non-zero components $R_{ijks}=R(X_i,X_j,X_k,X_s)$ of the curvature tensor R as follows:

$$
(3.11)\qquad \begin{array}{l}
-\frac{1}{2}R_{1221}=-R_{1341}=-R_{2342}=R_{3123}=\frac{1}{2}R_{3443}=R_{4124}=\lambda^2+\mu^2,\\
R_{1331}=R_{1441}=R_{2332}=R_{2442}=-R_{1324}=-R_{1423}=\lambda^2-\mu^2,\\
R_{1231}=R_{1241}=R_{2132}=R_{2142}\\
=-R_{3143}=-R_{3243}=-R_{4134}=-R_{4234}=2\lambda\mu.
\end{array}
$$

Then, by (1.7) and (3.11) we get the components $\rho_{ij}=\rho(X_i,X_j)$ of the Ricci tensor and the values of the scalar curvatures τ and τ^*:

$$
(3.12)\qquad \begin{array}{ll}
\rho_{11}=\rho_{22}=-4\lambda^2, & \rho_{33}=\rho_{44}=-4\mu^2,\\
\rho_{12}=\rho_{34}=-2(\lambda^2+\mu^2), & \rho_{13}=\rho_{14}=\rho_{23}=\rho_{24}=4\lambda\mu,\\
\tau=-8(\lambda^2-\mu^2), & \tau^*=16\lambda\mu.
\end{array}
$$

Let us consider the characteristic 2-planes α_{ij} spanned by the basic vectors $\{X_i,X_j\}$: totally real 2-planes - α_{12}, α_{14}, α_{23}, α_{34} and holomorphic 2-planes - α_{13}, α_{24}. By (1.10) and (3.11) for the sectional curvatures of the holomorphic 2-planes we obtain

$$
(3.13)\qquad k(\alpha_{13})=k(\alpha_{24})=-(\lambda^2-\mu^2).
$$

Then it is valid

Theorem 3.2. *The manifold (G,J,g) is of constant holomorphic sectional curvature.*

Using (1.9), (3.11) and (3.12) for the essential non-zero components $W_{ijks} = W(X_i, X_j, X_k, X_s)$ of the Weyl tensor W we get:

$$
\begin{aligned}
&\tfrac{1}{2}W_{1221} = W_{1331} = W_{1441} = W_{2332} = W_{2442} = \tfrac{1}{2}W_{3443} \\
&= -\tfrac{1}{3}W_{1324} = -\tfrac{1}{3}W_{1423} = \tfrac{1}{3}(\lambda^2 - \mu^2).
\end{aligned} \tag{3.14}
$$

Finally, by (1.9), (3.10), (3.12), (3.13) and (3.14) we establish the truthfulness of

Theorem 3.3. *The following conditions are equivalent*:

(i) (G,J,g) *is isotropic Kählerian*;

(ii) $|\lambda| = |\mu|$;

(iii) $\tau = 0$;

(iv) (G,J,g) *is of zero holomorphic sectional curvature*;

(v) *the Weyl tensor vanishes.*

(vi) $R = \frac{1}{2}\psi_1(\rho)$.

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